ON LINEARIZED SUPERSONIC FLOWS IN LAVAL NOZZLES

(O LINEARIZOVANYKH SVERKHZVUKOVYKH TECHENIIAKH V SUPLAKH LAVALIA)

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In this paper we investigate the supersonic gas flows near the axis of Laval nozzles, which are described by linearized hyperbolic equations of the second order.

An expression will be obtained for the determination of the form of the stream lines as a function of the distribution of Mach numbers along the axis of the nozzle. The effect of discontinuities in the initial values on the behavior of the stream lines will be investigated.

These solutions do not apply in the region of transition through the velocity of sound.

1. The equations of the characteristics of supersonic axially symmetric irrotational flows (Fig. 1) in the characteristic coordinates (η, ξ) have the following form:

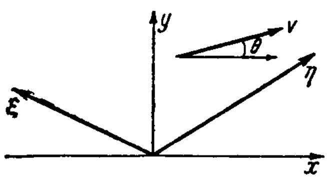

Fig. 1.

$$\frac{\partial\sigma}{\partial\eta} - \frac{\partial\theta}{\partial\eta} - \frac{\sin\theta\sin\mu}{y} = 0, \qquad \frac{\partial\sigma}{\partial\xi} + \frac{\partial\theta}{\partial\xi} + \frac{\sin\theta\sin\mu}{y} = 0 \qquad \left(\sigma(M) = \int_{v*}^{v} \cot\mu \frac{\partial v}{v}\right) \tag{1.1}$$

where μ is the Mach angle, M is the Mach number, v_* is the critical velocity and θ is the angle between the velocity vector and the x-axis.

The characteristic coordinates are expressed in terms of Cartesian coordinates by the following relationships:

$$\frac{\partial x}{\partial \eta} = \cos(\mu + \theta), \qquad \frac{\partial x}{\partial \xi} = -\cos(\mu - \theta), \qquad \frac{\partial y}{\partial \eta} = \sin(\mu + \theta), \qquad \frac{\partial y}{\partial \xi} = \sin(\mu - \theta) \tag{1.2}$$

To linearize the equations let it be assumed that the angles of inclination of the stream lines are small

$$\theta = o(1) \tag{1.3}$$

and in the equations we shall neglect small terms of higher order. This condition applies in a certain neighborhood of the axis of the nozzle.

Equations (1.1) show that in any characteristic triangle, $\sigma - \sigma_0$ and θ are quantities of the same order of smallness (here σ_0 is the value at a given point of the triangle). Hence it follows that within the linear approximation $\sigma = \sigma_0 = \text{const}$ and $\mu = \mu(\sigma) = \text{const}$.

In linearized form equations (1.1) become

$$\frac{\partial \sigma}{\partial \eta} - \frac{\partial \theta}{\partial \eta} - \frac{\theta \sin \mu}{y} = 0, \qquad \frac{\partial \sigma}{\partial \xi} + \frac{\partial \theta}{\partial \xi} + \frac{\theta \sin \mu}{y} = 0 \tag{1.4}$$

and the relationships between the coordinates are

$$\frac{\partial x}{\partial \eta} = \cos \mu, \qquad \frac{\partial x}{\partial \xi} = -\cos \mu, \qquad \frac{\partial y}{\partial \eta} = \sin \mu, \qquad \frac{\partial y}{\partial \xi} = \sin \mu \tag{1.5}$$

The characteristics of both families are straight lines having constant angles μ and $-\mu$ with the x-axis. After elimination of σ from (1.4), we obtain the differential equation for the determination of θ in Cartesian coordinates

$$\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial}{\partial y} \frac{\theta}{y} - c^2 \frac{\partial^2 \theta}{\partial x^2} = 0, \qquad (c = \cot \mu) \tag{1.6}$$

To solve equation (1.6) it is necessary that $\partial\theta/\partial y$ be given at $y = 0$. The second initial condition, derived from the condition of symmetry, is $\theta(x,\ 0) = 0$. Usually in calculations of Laval nozzles the distribution of M numbers (or $\sigma(M)$) along the axis of the nozzle is given. The connection between $(\partial\theta/\partial y)_{y=0}$ and $\sigma(x,\ 0)$ may be obtained from any of equations (1.4). We have

$$\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = \frac{c}{2}\left(\frac{\partial \sigma}{\partial x}\right)_{y=0} \tag{1.7}$$

2. Substitution of $u = \theta/y$ leads to the Darboux equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{3}{y}\frac{\partial u}{\partial y} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

To obtain solutions which are continuous at $y = 0$ it is necessary to assume, in order to account for the singularity in the second term

$$(\partial u / \partial y)_{y=0} = 0$$

This equation is one of the initial conditions. Therefore, only the second of the initial conditions may be chosen arbitrarily

$$u(x, 0) = \varphi(x) \quad \text{or} \quad u(x, 0) = (\partial\theta / \partial y)_{y=0} \tag{2.2}$$

3. We shall first find the solution U of equation (2.1), satisfying the simple initial condition

$$\varphi(x) = \tau(x, x_0) = \begin{cases} 1 & \text{for } x > x_0 \\ 0 & \text{for } x < x_0 \end{cases} \tag{3.1}$$

Equation (2.1) is homogeneous; therefore, we shall look for the solution U as a function of $z = (x - x_0)/y$. Substitution of this function into (2.1) leads to the ordinary differential equation

$$(c^2 - z^2)\, d^2U / dz^2 + z dU / dz = 0 \tag{3.2}$$

The boundary conditions for U have the form

$$U = \begin{cases} 1 & \text{for } x - x_0 = cy \text{ or } z = c \\ 0 & \text{for } x - x_0 = -cy \text{ or } z = -c \end{cases} \tag{3.3}$$

and the solution is

$$U = \frac{1}{2} + \frac{z}{c\pi}\sqrt{1 - \frac{z^2}{c^2}} + \frac{1}{\pi} \sin^{-1} \frac{z}{c} \qquad (-c \leqslant z \leqslant c) \tag{3.4}$$

$$U = 0 \quad \text{for } z \leqslant -c, \qquad U = 1 \quad \text{for } z \geqslant c$$

4. This function may be used for the construction of a solution in terms of arbitrary, but continuous initial values of $\varphi(x)$.

The continuous function $\varphi(x)$ may be represented in the form

$$\varphi(x) = \varphi(x - cy) + \int_{x-cy}^{\infty} \frac{d\varphi}{dx_0} \tau(x, x_0)\, dx_0 \tag{4.1}$$

because $\tau = 0$ for $x_0 > x$ according to (3.1). An increase in the initial value

$$d\varphi = \frac{d\varphi}{dx_0} \tau(x, x_0)\, dx_0$$

leads to the increase in the function at the point (x, y)

$$du = \frac{d\varphi}{dx_0} U\left(\frac{x - x_0}{y}\right) dx_0$$

The full value of u, obviously, is represented by equation

$$u = \varphi(x - cy) + \int_{x-cy}^{x+cy} \frac{d\varphi}{dx_0} U\left(\frac{x - x_0}{y}\right) dx_0 \tag{4.2}$$

because $U = 0$ for $x_0 > x + cy$. Hence, when integrating by parts and taking into account values of U on the characteristics (3.3), we obtain

$$u = \frac{1}{y} \int_{x-cy}^{x+cy} \varphi(x_0) \frac{dU}{dz} dx_0 \qquad \left(\frac{dU}{dz} = \frac{2}{c\pi} \sqrt{1 - \frac{(x - x_0)^2}{c^2 y^2}}\right) \tag{4.3}$$

It may be shown that equation (4.3) applies to all instances where the distribution of M numbers along the nozzle axis is continuous, i.e. to flows without strong discontinuities. Note that at points where the function $\delta(x)$ and its derivative $d\varphi/dx$ have discontinuities condition $\partial u/\partial y = 0$ for $y \to 0$ is not satisfied.

5. Returning to the function $\theta = uy$, we obtain as a solution of equation (1.6)

$$\theta = \int_{x-cy}^{x+cy} \varphi(x_0) \frac{dU}{dz} dx_0 \tag{5.1}$$

Consider the function θ for which

$$\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = \varphi(x) = \begin{cases} ax^n & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{5.2}$$

Let us assume the function $\varphi(x)$ at the point $x = 0$ to have a discontinuity of order n.

According to (5.1) the expression for θ in the region between characteristics $x = \pm\, cy$ has the form

$$\theta = \int_{0}^{x+cy} a x_0^{\,n} \frac{dU}{dz} dx_0 \tag{5.3}$$

As a consequence of linearizing (1.3) the stream lines correspond to $y \approx$ const; therefore relation (5.3) at $y = y_0$ represents the variation of the angle θ along a stream line $y \approx y_0$.

Let us introduce the new variable

$$X = x + cy \tag{5.4}$$

This substitution is equivalent to translation of the origin of the x coordinate to the point of intersection of the characteristic $x = -cy$ and the line $y = y_0$.

The expansion of dU/dz into a series in terms of $(X - x_0)/y_0$ has the form

$$\frac{dU}{dz} = \frac{2}{\pi c}\sqrt{\frac{2}{cy_0}}\sqrt{X - x_0}\left[1 + \ldots a_k \frac{(X - x_0)^k}{y_0{}^k} + \ldots\right] \tag{5.5}$$

Let us also expand θ into a series in terms of X/y_0 (5.6)

$$\theta = \frac{2}{\pi c}\sqrt{\frac{2}{cy_0}}\, aX^{n+1,5}\left(c_0 + \ldots + c_k \frac{X^k}{y_0{}^k} + \ldots\right), \quad c_i = a_i \int_0^1 \left(1 - \frac{x_0}{X}\right)^{0,5+i}\left(\frac{x_0}{X}\right)^n d\,\frac{x_0}{X}$$

For $n > -1$ all the coefficients c_i have meaning. When integrating with respect to X we obtain an expansion of the equation of the stream lines

$$y - y_0 = \frac{2}{\pi c}\sqrt{\frac{2}{cy_0}}\, aX^{n+2.5}\left[\frac{c_0}{n + 2.5} + \ldots + \frac{c_k X^k}{(n + k + 2.5)\, y_0{}^k} + \ldots\right] \tag{5.7}$$

Relation (5.7) shows that for a discontinuity in the initial function $\varphi(x)$ of order n, a discontinuity of order $n + 2.5$ in the stream line propagates along the characteristic $x = -cy$. When strong discontinuities are absent on the x-axis the order of the discontinuity in the stream line is greater than 1.5. To investigate the behavior of θ in front of the reflected characteristic $x = cy$ we introduce the change of variables

$$X_1 = cy_0 - x, \qquad x_1 = x_0 + X_1 \tag{5.8}$$

Expansion of dU/dz into a series in terms of x_1/y_0 has the form

$$\frac{dU}{dz} = \frac{2}{\pi c}\sqrt{\frac{2}{cy_0}}\sqrt{x_1}\left(1 + \ldots + a_k \frac{x_1{}^k}{y_0{}^k} + \ldots\right) \tag{5.9}$$

The expression for θ is transformed in the following manner:

$$\theta = \int_{X_1}^{2cy_0} a\,(x_1 - X_1)^n \frac{dU}{dz}\, dx_1 = \int_{X_1}^{2X_1} aX_1{}^n \left(\frac{x_1}{X_1} - 1\right)^n \frac{dU}{dz}\, dx_1 + \int_{2X_1}^{2cy_0} ax_1{}^n\left(1 - \frac{X_1}{x_1}\right)^n \frac{dU}{dz}\, dx_1 =$$

$$= aX_1{}^{n+1}\int_1^2 \left(\frac{x_1}{X_1} - 1\right)^n \frac{dU}{dz}\, d\,\frac{d_1}{X_1} + a\int_{2X_1}^{2cy_0} x_1{}^n\left(1 + \ldots + b_k \frac{X_1{}^k}{y_0{}^k} + \ldots\right)\frac{dU}{dz}\, dx_1 \tag{5.10}$$

Let N be a positive integer or zero. Then (5.10) may be represented as follows:

for $n \neq -0.5 + N$

$$\theta = y_0{}^{n+1}\left(A_0 + \ldots + A_k \frac{X_1{}^k}{y_0{}^k} + \ldots\right) + \frac{X_1{}^{n+1.5}}{y_0{}^{0.5}}\left(B_0 + \ldots + B_k \frac{X_1{}^k}{y_0{}^k} + \ldots\right) \tag{5.11}$$

for $n = 0.5 + N$ (5.12)

$$\theta = y_0^{n+1}\left(c_0 + \ldots + c_k \frac{X_1^k}{y_0^k} + \ldots\right) + \frac{X_1^{n+1.5}}{y_0^{0.5}} \ln \frac{X_1}{y_0}\left(D_0 + \ldots + D_k \frac{X_1^k}{y_0^k} + \ldots\right)$$

When differentiating the latter relation we see that the curvature of the stream lines $k = \partial\theta/\partial x$ for $n = -0.5$ on the reflected characteristic has a logarithmic singularity. An analogous result was obtained in [3] for the derivatives of the velocity for this particular case. Behind the reflected characteristic $x = cy$ we have for θ

$$\theta = \int_0^{X_2} a X_2^n \left(1 + \frac{x_2}{X_2}\right)^n \frac{dU}{dz}\, dx_2 + \int_{X_2}^{2cy_0} a x_2^n \left(1 + \frac{X_2}{x_2}\right)^n \frac{dU}{dz}\, dx_2 \qquad \begin{pmatrix} X_2 = x - c\,y_0 \\ x_2 = x_0 - X_2 \end{pmatrix} \quad (5.13)$$

where

$$\frac{dU}{dz} = \frac{2}{\pi c}\sqrt{\frac{2}{cy_0}}\sqrt{x_2}\left(1 + \ldots + a_k \frac{x_2^k}{y_0^k} + \ldots\right) \quad (5.14)$$

This expression transforms into a polynomial of order n, when n is a real number or zero, otherwise it has the form analogous to (5.11) or (5.12).

With the help of formulas (5.3) and (5.1) and also by means of integrating and differentiating them we obtain the following relations for the angle θ, the displacement of the stream line $y - y_0$ and for the curvature of the stream line k which are valid in the entire region:

$$\theta = y_0^{n+1} F_1\left(\frac{x}{y_0}\right), \qquad y - y_0 = y_0^{n+2} F_2\left(\frac{x}{y_0}\right), \qquad k = y_0^n F_3\left(\frac{x}{y_0}\right) \quad (5.15)$$

6. As an example of the application of equation (5.1) let us calculate the shape of the Laval nozzle, which transforms a radial flow into parallel flow.

The distance from the "source" to the point on the x-axis of transition from radial to parallel flow shall be denoted by R and the outlet radius of the nozzle by r. Obviously, $r/R = \theta_a$ is the angle of the wall of the nozzle in the radial region. Let us locate the origin of the x-axis at the point of contact of the two regions. Obviously

$$\varphi(x) = \begin{cases} 1/(R + x) & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases} \quad (6.1)$$

The quantity $1/(R + x_0)$ may be represented as follows:

$$\frac{1}{R + x_0} = \frac{1}{R + x}\left[1 + \frac{y}{R + x} z + \frac{y}{(R + x)^2} z^2 + \ldots\right]$$

Since $y/(R = x) = 0(\theta)$ only the first term $1/(R + x)$ in this expansion needs to be retained. Using (5.1) we obtain the expression for θ

$$\theta = \int_{x-cy}^{0} \frac{1}{R+x_0} \frac{dU}{dz} dx_0 = \frac{y}{R+x}\left(\frac{1}{2} - \frac{x}{\pi cy}\sqrt{1-\frac{x^2}{c^2y^2}} - \frac{1}{\pi} \sin^{-1}\frac{x}{cy}\right) \quad (6.2)$$

To calculate the equation of the stream line $y \approx r$ approximately this equation must be integrated with $y = r$. Here we can assume

$$\frac{1}{R+x} = \frac{1}{R}\left(1 - \frac{x}{R} + \frac{x^2}{R^2} + \ldots\right) \approx \frac{1}{R} \qquad \left(\frac{x}{R} = \frac{r}{R}\frac{x}{r} = O(\theta)\right)$$

That equation has the form

$$y - r_0 = \frac{xr}{2R} - \frac{xr}{\pi R} \sin^{-1}\frac{x}{cr} - \frac{cr^2}{\pi R}\sqrt{1-\frac{x^2}{c^2r^2}} + \frac{cr^2}{3\pi R}\left(1-\frac{x^2}{c^2r^2}\right)^{3/2} \qquad (-cr \leqslant x \leqslant cr) \quad (6.3)$$

7. When eliminating the function θ from equation (1.4) by using the coordinates x and y, we obtain

$$\frac{\partial^2\sigma}{\partial y^2} + \frac{1}{y}\frac{\partial\sigma}{\partial y} - c^2\frac{\partial^2\sigma}{\partial x^2} = 0 \quad (7.1)$$

From the initial conditions

$$\sigma(x, 0) = \psi(x), \qquad (\partial\sigma / \partial y)_{y=0} = 0 \quad (7.2)$$

the solution of equation (7.1) has the form

$$\sigma = \int_{x-cy}^{x+cy} \psi(x_0) \frac{1}{\pi cy} \frac{dx_0}{\sqrt{1-(x-x_0)^2/c^2y^2}}, \qquad \varphi(x) = \frac{c}{2}\frac{d\psi}{dx} \quad (7.3)$$

8. For comparison with the relationships obtained above we give (without derivation) some results for the case of plane flow. Equations for θ and σ have the form

$$\frac{\partial^2\theta}{\partial y^2} - c^2\frac{\partial^2\theta}{\partial x^2} = 0, \qquad \frac{\partial^2\sigma}{\partial y^2} - c^2\frac{\partial^2\sigma}{\partial x^2} = 0 \quad (8.1)$$

and the solution (D'Alembert's) of these equations is

$$\theta = \frac{1}{2c}\int_{x-cy}^{x+cy} \varphi(x_0)\, dx_0, \qquad \sigma = \frac{\psi(x+cy) + \psi(x-cy)}{2}, \qquad \varphi(x) = \left(\frac{\partial\theta}{\partial y}\right)_{y=0}, \quad (8.2)$$

$$\psi(x) = \sigma(x, 0), \qquad \theta(x, 0) = 0, \qquad (\partial\sigma / \partial y)_{y=0} = 0$$

When the values of σ are continuous on the axis, which is the case in the absence of strong discontinuities on the axis, the stream lines have discontinuities of an order higher than the first, i.e. they do not have corner points.

The order of the discontinuities of stream lines, propagating along the characteristics is greater by two than the order of the discontinuities of $\varphi(x)$. In particular, in the case of the simplest initial conditions (3.1) the stream lines in the region $-cy \leqslant x - x_0 \leqslant cy$ represent the chords of curved segments, which become tangent lines on the characteristics $x - x_0 = \pm cy$.

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